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Gibbs state for some class of meromorphic functions

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ABSTRACT

This paper is a continuation of our earlier works [1,2] on the fractal structure of expanding and subexpanding meromorphic functions of the form $F = H \circ \exp \circ Q$, where H and Q are non-constant rational maps. Under some assumptions on the forward trajectories of asymptotic values of F we define a class of summable potentials for the maps f of the punctured cylinder induced by F. We prove the existence and uniqueness of Gibbs states for these potentials.

1. INTRODUCTION

We consider the class $\mathcal{F}$ of transcendental meromorphic functions $F(z) : \mathbb{C} \to \overline{\mathbb{C}}$ of the form

$$F(z) = H \circ \exp(z), \tag{1.1}$$

where H is a non-constant rational map. The set of singularities $\mathrm{Sing}(F^{-1})$ consists of finitely many critical values and two asymptotic values $H(0)$, $H(\infty)$. We assume that $H(0) \neq \infty$, $H(\infty) \neq \infty$ and

$$\eta := \mathrm{dist}_\chi\big(P_1(F), J_F\big) > 0, \tag{1.2}$$

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where χ is a chordal metric,

$$P_1(F) := \overline{\Theta^+\big(\{H(0), H(\infty)\}\big)} = \overline{\bigcup_{n=0}^{\infty} F^n\big(\{H(0), H(\infty)\}\big)}.$$

Thus F is a non-entire function. The subclass of these functions we denote by $\mathcal{F}_1$. Since $F(z)$ is $2\pi i$-periodic, we consider it rather on the cylinder than on $\mathbb{C}$. So let $\mathcal{P}$ be the quotient space (the cylinder)

$$\mathcal{P} = \mathbb{C}/\sim,$$

where $z_1 \sim z_2$ if and only if $z_1 - z_2 = 2k\pi i$ for some $k \in \mathbb{Z}$. Let $\pi : \mathbb{C} \to \mathcal{P}$ be the canonical projection. The function F projects down to a holomorphic map

$$f : \mathcal{P} \setminus \pi\big(F^{-1}(\infty)\big) \mapsto \mathcal{P}$$

so that $f \circ \pi = \pi \circ F$. The Julia set J_f of f is defined to be

$$J_f := \pi(J_F \cap \mathbb{C}).$$

Notice that if $F \in \mathcal{F}_1$ there exists $M > 0$ such that

$$(1.3) \qquad -M < \operatorname{Re} J_f < M.$$

We introduce on $\mathcal{P}$ a class of summable potentials φ and a transfer operator $\mathcal{L}_\varphi$ which acts on the Banach space $C(J_f)$ of continuous functions on J_f. For every $z \in \mathcal{P}$ we define

$$\begin{aligned} P_z(\varphi) &= \limsup_{n\to\infty} \frac{1}{n} \log \mathcal{L}_\varphi^n 1(z) \\ &= \limsup_{n\to\infty} \frac{1}{n} \log \sum_{y \in f^{-1}(z)} \exp\big(\varphi(y) + \cdots + \varphi\big(f^{n-1}(y)\big)\big). \end{aligned}$$

We prove the following result.

Theorem A. *Let $F \in \mathcal{F}_1$, $f = \pi(F)$. Assume that φ is an summable potential such that* $\sup \varphi < \sup_{z \in J_f} P_z(\varphi)$. *Then*:

1. *The limit $P(\varphi) = \lim_{n\to\infty} \frac{1}{n} \log \mathcal{L}_\varphi^n 1(z)$ exists and is independent of $z \in J_f$.*
2. *There exists a unique* $\exp(P(\varphi) - \varphi)$*-conformal measure m_φ on J_f and a unique Borel probability f-invariant measure μ_φ absolutely continuous with respect to the measure m_φ. The measure μ_φ is in fact equivalent to m_φ, $h = \frac{d\mu_\varphi}{dm_\varphi}$, the normalized* (*so that $\int \rho \, d\mu_\varphi = 1$*) *fixed point of the Perron–Frobenius operator $\hat{\mathcal{L}}_\varphi$, and is called the Gibbs state of the summable potential φ.*

In Section 2 we recall the concept of Walters expanding conformal maps and show its application to expanding maps in $\mathcal{F}$. In Section 3 we collect the results on existence of conformal measures for functions $f \in \mathcal{F}$ such that at least one asymptotic value is eventually mapped onto ∞. In Section 4 we introduce a class of summable potentials for the maps of the punctured cylinder induced by the functions from the subclass $\mathcal{F}_1$. Finally in Section 5 we prove Theorem A.

2. GIBBS STATES FOR EXPANDING MAPS IN THE CLASS $\mathcal{F}$

In [1] there was described the fractal structure of expanding meromorphic functions of the form

$$F(z) = H \circ \exp \circ Q(z), \tag{2.1}$$

where H and Q are non-constant rational functions. Before we state the main result proved for these functions in [1], we briefly remind the concept of Walters expanding conformal maps defined there.

Let X_0 be an open and dense subset of a compact metric space X endowed with a metric d. We call a continuous map $T : X_0 \to X$ Walters expanding provided that the following conditions are satisfied:

(a) The set $T^{-1}(x)$ is at most countable for each $x \in X$.
(b) There exists $\delta > 0$ such that for every $x \in X$ and every $n \geqslant 0$, $T^{-n}(B(x, 2\delta))$ can be written uniquely as a disjoint union of open sets $\{B_y(x)\}_{y \in T^{-n}(x)}$ such that $y \in B_y(x)$ and $T^n : B_y(x) \to B(x, 2\delta)$ is a homeomorphism from $B_y(x)$ onto $B(x, 2\delta)$. The corresponding inverse map from $B(x, 2\delta)$ to $B_y(x)$, $y \in T^{-n}(x)$, will be denoted by T_y^{-n}.
(c) There exist $\lambda > 1$ and $n \geqslant 1$ such that for every $x \in X$, every $y \in T^{-n}(x)$ and all $z_1, z_2 \in B_y(x)$

$$d\big(T^n(z_1), T^n(z_2)\big) \geqslant \lambda d(z_1, z_2).$$

(d) $\forall \epsilon > 0\, \exists s \geqslant 1\, \forall x \in X\ T^{-s}(x)$ is ϵ-dense in X.

A function $\varphi : X_0 \to \mathbb{R}$ is called dynamically Hölder if there exists $\beta > 0$ and $L > 0$ such that for every $n \geqslant 1$, every $x \in X$ and every $y \in T^{-n}(x)$, the restriction $\varphi|_{T_y^{-n}(B(x,\delta))}$ is Hölder continuous with exponent $\beta > 0$ and constant L. For every $n \geqslant 1$ put

$$S_n\varphi(x) = \sum_{j=0}^{n-1} \varphi \circ T^j(x).$$

The function $\varphi : X_0 \to \mathbb{R}$ is called summable if

$$\sup_{x \in X} \left\{ \sum_{y \in T^{-1}(x)} \exp\big(\varphi(y)\big) \right\} < \infty.$$

Given $x \in X$, similarly as before, we set

$$P_x(\varphi) = \limsup_{n\to\infty} \frac{1}{n} \log \sum_{y\in T^{-n}(x)} \exp\big(S_n\varphi(y)\big).$$

It is not difficult to prove that if $\varphi: X_0 \to \mathbb{R}$ is dynamically Hölder, then $P_x(\varphi) = P_y(\varphi)$ for all $x, y \in X$. The common value is called the topological pressure of φ with respect to T and is denoted by $P(\varphi)$. We should notice that $P(\varphi) < \infty$ if and only if φ is summable.

Walters Theorem. [9] *If $T: X_0 \to X$ is a Walters expanding map and $\varphi: X_0 \to \mathbb{R}$ is a dynamically Hölder summable function, then there exist m_φ and μ_φ, Borel probability measures on X such that*

(a) $\forall n \geqslant 1$, $\forall x \in X$, $\forall y \in T^{-n}(x)$ *and for every Borel set* $A \subset T_y^{-n}(B(x,\delta))$

$$m_\varphi\big(T^n(A)\big) = \int_A e^{P(\varphi)-S_n(\varphi)}\, dm_\Phi,$$

(b) μ_φ *is T-invariant which means that* $\mu_\varphi \circ T^{-1} = \mu_\varphi$, *ergodic and equivalent to* m_φ *with continuous Radon–Nikodym derivative bounded away from zero and infinity.*

The reader familiar with thermodynamic formalism, may notice that the property (a) means that the measure m_φ is an eigenmeasure of the operator dual to the appropriate Perron–Frobenius operator $\mathcal{L}_\varphi$ with eigenvalue $e^{P(\varphi)}$.

A Walters expanding map $T: X_0 \to X$ is called conformal if $X \subset \mathbb{C}$ and if for every $x \in X$, every $n \geqslant 1$ and every $y \in T^{-n}(x)$ the inverse map $T_y^{-n}: B_X(x, 2\delta) \to X_0$ has a (unique) holomorphic extension to the ball $B_{\mathbb{C}}(x, 2\delta)$. This extension will be denoted by the same symbol T_y^{-n}. For a Walters expanding conformal map T we consider a potential $\varphi_t: X_0 \to \mathbb{R}$, $t \geqslant 0$, given by the formula

$$\varphi_t(x) = -t \log\big|T'(x)\big|.$$

It immediately follows from Koebe's distortion theorem that each function φ_t is dynamically Hölder with the Hölder exponent $1/3$. Following [6] we define θ_T to be the infimum of all $t \geqslant 0$ for which the function φ_t is summable. Due to Proposition 2.4 in [1],

$$\theta_T = \inf\big\{t \geqslant 0\colon\ P(\varphi_t) < \infty\big\}.$$

Then the function $P: (\theta_T, \infty) \to \mathbb{R}$ is convex, continuous, strictly decreasing and $\lim_{t\to+\infty} P(t) = -\infty$. Let

$$h_T = h = \inf\big\{t\colon\ P(t) \leqslant 0\big\}.$$

Obviously $h_T \geqslant \theta_T$. Following the terminology of [6] the map T is called regular if $P(h) = 0$, strongly regular if there exists $t \geqslant 0$ such that $0 < P(t) < \infty$ and hereditarily regular if $P(\theta_T) = \infty$. Then each strongly regular map is regular and each hereditarily regular map is strongly regular. If T is regular, then $m = m_{\varphi_h}$ is called the h-conformal measure for T. T-invariant measure μ equivalent to m is called Gibbs state corresponding to a potential $\varphi_t = -t \log|T'|$. Define

$$X_\infty = \bigcap_{n \geqslant 0} T^{-n}(X_0).$$

Assume that T is a Walters expanding conformal map, then:

1. $\mathrm{HD}(X_\infty) \leqslant h$. If additionally T is strongly regular, then $\mathrm{HD}(X_\infty) = h$ and, in particular, $\mathrm{HD}(X_\infty) > \theta_T$.
2. If T is a regular Walters expanding conformal map, then $\mathrm{H}^h(X_\infty) < \infty$ and $\mathrm{P}^h(X_\infty) > 0$. In addition, $\mathrm{H}^h \ll m_h$ and $m_h \ll \mathrm{P}^h$,

where HD denotes the Hausdorff dimension, H^h and P^h are respectively h-dimensional Hausdorff and packing measures. All these results were proved in [1].

In [1] the theory of Walters expanding conformal maps was applied to the functions

$$\tilde{F}(z) = \exp \circ\, Q \circ H(z), \quad z \in \overline{\mathbb{C}}, \tag{2.2}$$

where H and Q were defined in (2.1). Note that the maps F and $\tilde{F}$ are semiconjugate by H and $\exp(Q)$.

Analogously as in [3] we say that F is a Barański map if the following conditions are satisfied:

(1) $J_F \cap \overline{\bigcup_{n=0}^{\infty} f^n(\mathit{Crit}(F) \cup \mathit{Asymp}(F))} = \emptyset$,
(2) if $a \in \mathit{Crit}(Q)$, then $\exp(Q(a))$ is not a pole of H,
(3) if H has a multiple pole, then $Q(\infty) \neq \infty$,

where $\mathit{Asymp}(F)$ denotes the set of asymptotic values of F. It is easy to check that F defined in (2.1) is expanding if and only if F is a Barański map. The map $\tilde{F}$ is called a post-Barański map. It was proved in [1] that for post-Barański map there exists $0 < \kappa < \infty$ such that

$$J_{\tilde{F}} \subset \{z\colon\ e^{-\kappa} \leqslant |z| \leqslant e^{\kappa}\}$$

and $\tilde{F}$ is a Walters expanding conformal map. It leads to the main results of [1].

Proposition 2.1. *Assume that F is a Barański map. Let $h = \mathrm{HD}(J_F)$. Then there exists a unique Gibbs state μ equivalent to a conformal measure m_h corresponding to a summable potential $\varphi_t = -t \log|F'|$. Moreover,*

(a) *if $h < 1$, then $0 < \mathrm{P}^h(J_F) < \infty$ and $\mathrm{H}^h(J_F) = 0$,*

(b) *if* $h = 1$, *then* $0 < \mathrm{P}^h(J_F)$, $\mathrm{H}^h(J_F) < \infty$,
(c) *if* $h > 1$, *then* $0 < \mathrm{H}^h(J_F) < \infty$ *and* $\mathrm{P}^h(J_F) = \infty$,

where the Hausdorff measure and packing measure are defined by means of spherical metric.

If $Q(z) = z$ then Proposition 2.1 implies the following corollary.

Corollary 2.2. *Let F be a Barański map, $h = \mathrm{HD}(J_F)$. Then there exists a unique Gibbs state μ equivalent to a conformal measure m_h corresponding to a summable potential $\varphi_t = -t \log|F'|$.*

3. CONFORMAL MEASURES FOR MAPS WITH AN ASYMPTOTIC VALUE EVENTUALLY MAPPED ONTO A POLE

We begin this section with recalling the properties of those functions in the class $\mathcal{F}$ for which at least one asymptotic value is mapped onto ∞. This class of maps was studied in [7,8,2]. Let

$$I_q(F) := \Big\{z \in J_F\colon \lim_{n\to\infty} F^{kq}(z) = \infty\Big\}, \quad q \geqslant 1.$$

First we quote the result proved in [7].

Proposition 3.1. *Let $F \in \mathcal{F}$ and at least one asymptotic value is eventually mapped onto a pole. Then* $\mathrm{HD}(I_q(F)) = 2$. *It follows that* $\mathrm{HD}(J_F) = 2$.

Now we define a new subclass $\mathcal{F}_2$ of $\mathcal{F}$. The function $F \in \mathcal{F}_2$ if both asymptotic values $H(0)$ and $H(\infty)$ are eventually mapped onto ∞ and

$$\mathrm{dist}_\chi\big(P_2(F), J_F\big) > 0,$$

where $P_2(F) = \overline{\Theta^+(\mathrm{Sing}(F^{-1}) \setminus \Theta^+(H(0), H(\infty))}$. As before $f := \pi \circ F$. Then J_f^r is the set of points whose trajectory returns infinitely often to some compact set whose intersection with the postsingular set is empty.

Proposition 3.2. *Let $F \in \mathcal{F}_2$, $f = \pi \circ F$. Then*:

1. $1 < h := \mathrm{HD}(J_f^r) < 2$.
2. *There exists h-conformal measure m on J_f for f such that m is atomless and* $m(J_f) = 1$.
3. *If m' is a t-conformal probabilistic measure on J_f for some $t > 1$ then $m' = m$.*
4. *f is ergodic with respect to the measure m.*

For proof of Proposition 3.2 see [8]. Note that in this case the exponent of conformal measure $h \neq \mathrm{HD}(J_f)$. To the class $\mathcal{F}_2$ belong the maps $F_\lambda(z) = \lambda(1 - \exp(-2z))^{-1}, \lambda > 0$, considered in [2], where Proposition 3.2 was proved independently for those maps. Moreover, it was shown in [2] the following proposition.

Proposition 3.3. *Let* $F_\lambda(z) = \lambda(1 - \exp(-2z))^{-1}, \lambda > 0,\ f_\lambda = \pi \circ F_\lambda,\ h_\lambda = \mathrm{HD}(J^r_{f_\lambda})$. *Then*:

1. $0 < \mathrm{H}^{h_\lambda}(J^r_{f_\lambda}) < \infty$.
2. P^{h_λ} *is locally infinite at every point of* $J^r_{f_\lambda}$.
3. *There exists a unique Borel probability* f_λ*-invariant measure* μ_λ *on* $J^r_{f_\lambda}$ *absolutely continuous with respect to* H^{h_λ}, μ_λ *is ergodic.*

4. GIBBS STATES FOR MAPS IN THE CLASS $\mathcal{F}_1$

Let $\mathcal{P}$ be the cylinder and $\pi : \mathbb{C} \to \mathcal{P}$ be the canonical projection. The function $F \in \mathcal{F}_1$ projects down to a holomorphic map

$$f : \mathcal{P} \setminus \pi\big(F^{-1}(\infty)\big) \mapsto \mathcal{P}$$

so that $f \circ \pi = \pi \circ F$ i.e. the following diagram commutates:

$$\begin{array}{ccc} \mathbb{C} \setminus B_0 & \xrightarrow{F} & \mathbb{C} \\ \pi \big\downarrow & & \big\downarrow \pi \\ \mathcal{P} \setminus B & \xrightarrow{f} & \mathcal{P}, \end{array} \tag{4.1}$$

where $B_0 = F^{-1}(\infty)$ and $B = \pi(B_0)$. We remind that if $F \in \mathcal{F}_1$ then J_f is a compact subset of $\mathcal{P}$ (compare (1.3)). To find a Gibbs state we consider a Perron–Frobenius operator which in our context take on the form

$$\mathcal{L}_\varphi g(z) = \sum_{y \in f^{-1}(z)} e^{\varphi(y)} g(y).$$

Notice that the series defining the Perron–Frobenius operator $\mathcal{L}_\varphi$ is infinite and in order to make it well-defined and bounded on the Banach space $C(J_f)$ of continuous functions on J_f, one should demand that with a universal constant $C > 0$

$$\mathcal{L}_\varphi(\mathbb{1}) = \sum_{y \in f^{-1}(z)} e^{\varphi(y)} \leqslant C$$

for all $z \in J_f$. Fix $z_0 \in \pi^{-1}(z)$. Then $y \in f^{-1}(z)$ if and only if there exists $k \in \mathbb{Z}$ such that $f_0(y) = z_0 + 2k\pi i$, where $f_0 : \mathcal{P} \to \mathbb{C}$ is the auxiliary map defined by the canonical projection π i.e. $\pi \circ f_0 = f$. Therefore

$$\mathcal{L}_\varphi(\mathbb{1})(z) = \sum_{y \in f^{-1}(z)} e^{\varphi(y)} = \sum_{k \in \mathbb{Z}} \sum_{y \in f_0^{-1}(z_0 + 2k\pi i)} e^{\varphi(y)}.$$

If $|k|$ is big, then ${f_0}^{-1}(z_0 + 2k\pi i)$ is near the pole b of $f_0 : \mathcal{P} \to \mathbb{C}$, but

$$z_0 + 2k\pi i = f_0(y) = \frac{G_b(y)}{(y-b)^{q_b}}$$

with G_b, a holomorphic function defined near b, such that $G_b(b) \neq 0$ and q_b – the multiplicity of the pole b, $q_b \geqslant 1$. Since the set of poles $B \subset \mathcal{P}$ is finite the series

$$\sum_{b \in B} \sum_{y \in {f_0}^{-1}(z_0+2k\pi i)} |z_0 + 2k\pi i|^{-(1+\epsilon_b)}$$

converges with an arbitrarily chosen $\epsilon_b > 0$. Trying to apply the comparison test, we would therefore require that with same constant $L > 0$

$$\exp\bigl(\varphi(y)\bigr) \leqslant L|z_0 + 2k\pi i|^{-(1+\epsilon_b)} = L\left(\frac{|y-b|^{q_b}}{|G_b(y)|}\right)^{1+\epsilon_b}$$

for all poles $b \in B$ and all $y \in {f_0}^{-1}(z_0 + 2k\pi i)$ being close to b. Or equivalently,

$$\varphi(y) \leqslant \log L - (1+\epsilon_b)\log\bigl|G_b(y)\bigr| + (1+\epsilon_b)q_b \log|y-b|$$

near b. This inequality suggests us that we deal with the class of potentials $\varphi : J_f \to \mathbb{C}$, called in the sequel summable, satisfying the following two conditions.

(a) For any open set V containing B, φ is a Hölder continuous on $J_f \setminus V$.
(b) For every pole $b \in B$ there are $\epsilon_b > 0$ and Hölder continuous function H_b such that $\varphi(z) = H_b(z) + (1+\epsilon_b)q_b \log|z-b|$ on a sufficiently small neighborhood of b.

Note that the potential $\varphi_t(z) = -\log|f'(z)|$ do not need to be summable since we do not assume that $f_{|J_f}$ is expanding. In particular the critical points might belong to J_f.

Given a measurable function $\psi : J_f \to [0, +\infty]$, a Borel probability measure m on J_f is said to be ψ-conformal if and only if $m(J_f) = 1$ and

$$m\bigl(f(A)\bigr) = \int_A \psi \, dm$$

for every Borel set $A \subset J_f$ such that $f_{|A}$ is one-to-one. Due to compactness of the Julia set J_f, it is much easier here to construct (generalized) conformal measures. Namely, the map

$$\nu \to \frac{\mathcal{L}_\varphi^* \nu}{\mathcal{L}_\varphi^* \nu(\mathbb{1})}$$

is continuous on the compact convex set of Borel probability measures on J_f. So, the Schauder–Tichonov theorem applies and we obtain a $\rho e^{-\varphi}$-conformal measure with some constant $\rho > 0$. Define

$$P_z(\varphi) = \limsup_{n\to\infty} \frac{1}{n} \log \sum_{y\in f^{-n}(z)} \exp\left(\sum_{j=0}^{n-1} \varphi \circ f^j(y)\right) = \limsup_{n\to\infty} \frac{1}{n} \log \mathcal{L}_\varphi^n(\mathbb{1})(z).$$

The key point to obtain all the results discussed below is a very detailed analysis of the behavior of the normalized Perron–Frobenius operator $\hat{\mathcal{L}}_\varphi = \rho^{-1}\mathcal{L}_\varphi$. Apart from (a) and (b) the third general assumption is that

$$\sup\{P_z(\varphi)\colon\ z \in J_f\} > \sup(\varphi).$$

Theorem 4.1. *For every $z \in J_f$, $P_z(\varphi)$ is the same, and the common value $P(\varphi)$, called the topological pressure of φ, is given by the following formula*

$$P(\varphi) = \lim_{n\to\infty} \frac{1}{n} \log \mathcal{L}_\varphi^n(z) = \lim_{n\to\infty} \frac{1}{n} \log \left\| \mathcal{L}_\varphi^n(\mathbb{1}) \right\|_\infty.$$

If μ is $te^{-\varphi}$-conformal f-invariant measure, then $\log(t) = P(\varphi)$.

Theorem 4.2. *There exists a unique* $\exp(P(\varphi) - \varphi)$*-conformal measure m_φ on J_f and a unique Borel probability f-invariant measure μ_φ absolutely continuous with respect to the measure m_φ. The measure μ_φ is in fact equivalent to m_φ, $h = \frac{d\mu_\varphi}{dm_\varphi}$, the normalized (so that $\int \rho\, d\mu_\varphi = 1$) fixed point of the Perron–Frobenius operator $\hat{\mathcal{L}}_\varphi$, and is called the Gibbs state of the summable potential φ.*

5. PROOF OF THEOREM A

The proofs of Theorem 4.1 and Theorem 4.2 are based on the proof of Theorem 1.1 from [5]. We are not going to repeat all the arguments used there. We only want to show a reader the main steps and explain why they work in our context.

Define $\mathcal{P}_M = \{z \in \mathcal{P}\colon\ -M < \operatorname{Re} z < M\}$, where M was defined in (1.3). Let $\|\cdot\|$ denote the sup-norm and $\|\varphi\|_E = \sup_{z\in E} |\varphi(z)|$, $E \subset \mathcal{P}_M$. For a simply-connected domain U we denote $distortion(U) = \frac{R}{r}$, where $R = \inf\{R > 0\colon\ U \subset D(0,R)\}$ and $r = \sup\{r > 0\colon\ U \supset D(0,r)\}$. Modulus of an annulus A is denoted by $mod(A)$.

We fix a point $z_0 \in \mathcal{P}_M$ for which

$$\log(\sigma) := \sup\varphi - P_{z_0}(\varphi) < 0$$

and let m_φ be a $\rho e^{-\varphi}$-conformal measure with $\rho = e^{P_{z_0}(\varphi)}$. Fix $\sigma < \lambda < 1$ and define

$$\alpha = \frac{\kappa d}{1-\sigma} + \frac{1}{\lambda - \sigma},$$

where d is the number of critical points of f_0. We also fix $m \geqslant 1$ such that $\alpha\sigma^m < 1$. Then we choose a topological disk $U \subset \mathcal{P}_M$ which does not contain critical values of f^m such that U is dense in $\mathcal{P}_M$ and $U \cap J_f \neq \emptyset$. We also assume that:

– there exists a lift U_0 of U such that $\pi : U_0 \to U$ is conformal, $U_0 \subset D(0,r)$ for a fixed $r > 0$,
– ∂U is piecewise smooth.

Then there exist well-defined inverse branches $h_j^m : U \to U_j^m$ of f^m, $j \in I_m$. The next lemma allows to control $distortion(U_j^m)$, $U_j^m = h_j^m(U)$.

Lemma 5.1. *There are constants $K \geqslant 1, \kappa \in \mathbb{N}$, with κ depending only on (the fixed) radius $r > 0$ and there are simply-connected domains V_j^m, $j \in I_m$, such that for all $j \in I_m$ the following holds*:

– $\overline{U_j^m} \subset V_j^m$,
– $mod(V_j^m \setminus \overline{U_j^m}) \geqslant \frac{1}{K}$,
– $distortion(U_j^m) \leqslant K$,
– *the family $\{V_j^m, j \in I_m\}$ is of multiplicity at most κ, i.e. any point $z \in \mathcal{P}_M$ is in at most κ sets V_j^m.*

Proof. The considered functions $F = H \circ \exp$ have only finitely many critical and asymptotic values. It follows from (1.2) that the forward trajectories of asymptotic values $H(0)$ and $H(\infty)$ can not approach the Julia set J_F, so the only possible singular values which belong or accumulate on J_F are critical values. Finally, f and f_0 have only finitely many critical points in the cylinder $\mathcal{P}$. All these argument show that the proof of Lemma 4.1 from [5] works in our case too. □

Now we split inverse branches (h_j^m) between these which shrinks exponentially and the others. The next lemma, proven in [5] (comp. Lemma 4.3), estimates the number of some 'bad branches'.

Lemma 5.2. *Let $0 < \lambda < 1$, $E_m = \emptyset$ and, for $n > m$, let E_n be the set of all $j \in I_n$ such that* $\mathrm{diam}(U_j^n) > k\lambda^{\frac{n-m}{2}}$. *Then E_n has at most $\lambda^{-(n-m)}$ elements for $n \geqslant m$.*

We define the index set J_n corresponding to the exponentially shrinking branches. Let $J_m = I_m$, since in view of Lemma 5.1 $E_m = \emptyset$. Assume that J_n was already defined for $n \geqslant m$. Then

$$J_{n+1} = \left\{ j \in I_{n+1} \colon \ f \circ h_j^{n+1} = h_i^n \text{ for some } i \in J_n \right\} \setminus E_{n+1}.$$

The other 'bad inverse branches' can be characterized as follows. See also Lemma 4.2 in [5].

Lemma 5.3. *In the setting of Lemma 5.1, there exist, for every $n \geqslant m$, holomorphic inverse branches $h_j^n : U \to U_j^n \subset \mathcal{P}_M$, $i \in I_n$, of f^n having the following properties*:

1. *for any $i \in I_{n+1}$ there exists $j \in I_n$ such that $f \circ h_i^{n+1} = h_j^n$,*

2. *there exists $K > 0$ such that, for all $n \geqslant m$ and $m \in I_n$ distortion$(U_i^n) \leqslant K$ and*

$$\frac{|(h_i^n \circ f^m)'(x)|}{|(h_i^n \circ f^m)'(x')|} \leqslant K$$

for all $x, x' \in f^{n-m}(U_i^n)$,
3. *Fix $z \in U$. For $n > m$, let $H_n(z)$ be the set of $y \in f^{-n}(z)$ such that there exists $j \in I_{n-1}$ with $h_j^{n-1}(z) = f(y)$ but $h_j^n(z) \neq y$ for all $j \in I_n$. Then $\sharp(H_n) \leqslant \kappa d$ for all $n > m$.*

Now we prove that the variation of the function $S_n\varphi$ along exponentially shrinking branches is uniformly bounded.

Lemma 5.4. *Let $0 < \lambda < 1$, $m \geqslant 1$ and U be a topological disk in $\mathcal{P}_M$ which does not contain critical values of f^m. Then there exists $A > 0$ (depending on λ, m and the Hölder constants of φ but not on U) such that for all $z, z' \in U$ and all $j \in J_n$, $n \geqslant m$, we have*

$$\left|S_n\varphi\big(h_j^n(z)\big) - S_n\varphi\big(h_j^n(z')\big)\right| \leqslant A.$$

Proof. We choose two neighbourhoods V_1, V_2 of the poles B such that $V_1 = \bigcup_{b\in B} D(b,r)$ and $V_2 \subset V_1$. Moreover, if $f^{-n}(W)$ is the inverse branch of f^n defined on some domain $W \subset \mathcal{P}_M$, then either $f_i^{-n}(W) \subset \mathcal{P}_M \setminus V_2$ or $f_i^{-n}(W) \subset B(b,r)$ for some pole b. Let $x_i = h_j^n(z)$, $x_i' = h_j^n(z')$, $x_{n-i} = f^i(x_n)$, $x'_{n-i} = f^i(x'_n)$, $0 \leqslant i \leqslant n$, where $j \in J_n$ and $z, z' \in D(b,r)$. Therefore

$$\left|x_i - x_i'\right| \leqslant K\lambda^{\frac{i-m}{2}} = K^*\lambda^{i/2}. \tag{5.1}$$

For the summable potential $\varphi(z) = H_b(z) + (1+\epsilon_b)q_b \log|z-b|$ we have

$$\begin{aligned} \left|\varphi(x_i) - \varphi\big(x_i'\big)\right| &\leqslant \left|H_b(x_i) - H_b\big(x_i'\big)\right| + (1+\epsilon_b)q_b\big|\log|x_i - b| - \log\left|x_i' - b\right|\big| \\ &\leqslant c_b\left|x_i - x_i'\right|^{\alpha} + (1+\epsilon_b)\log\left|\frac{x_i - b}{x_i' - b}\right|^{q_b}. \end{aligned} \tag{5.2}$$

Since

$$\left|\frac{x_i - b}{x_i' - b}\right|^{q_b} = \left|\frac{G_b(x_i)}{G_b(x_i')}\right|\left|\frac{x_i' + 2k\pi i}{x_i + 2k\pi i}\right| \tag{5.3}$$

for some $k \in \mathbb{Z}$, G_b is holomorphic and $G_b(z) \neq 0$ in $D(b, 2r)$, we obtain the following estimates:

$$\log\left|\frac{G_b(x_i)}{G_b(x_i')}\right| \leqslant \log\left(1 + \left|\frac{G_b(x_i) - G_b(x_i')}{G_b(x_i')}\right|\right) \leqslant c_b\left|x_i - x_i'\right|^{\alpha} \tag{5.4}$$

and

$$\log\left|\frac{x_{i-1}' + 2k\pi i}{x_{i-1}' + 2k\pi i}\right| \leqslant \log\left(1 + \left|\frac{x_{i-1}' - x_{i-1}'}{x_{i-1}' + 2k\pi i}\right|\right) \leqslant \left|x_{i-1}' + x_{i-1}\right|. \tag{5.5}$$

Now we apply (5.1) and (5.3)–(5.5) to (5.2)

$$\left|\varphi(x_i)-\varphi(x_i')\right| \leqslant c_b\left|x_i-x_i'\right|^{\alpha}+(1+\epsilon_b)q_b\left|\left(c_b\left|x_i-x_i'\right|^{\alpha}+\left|x_{i-1}-x_{i-1}'\right|\right)\leqslant C_b\lambda^{\frac{\alpha i}{2}}$$

for some $C_b>0$. On the other hand, if $x_i, x_i' \in \mathcal{P}_M \setminus V_2$ then

$$\left|\varphi(x_i)-\varphi(x_i')\right| \leqslant c\left|x_i-x_i'\right|^{\alpha} \leqslant cK^{*\alpha}\lambda^{\frac{\alpha i}{2}}.$$

Now we can sum up these estimations for all $0<i\leqslant n$ and get

$$\left|S_n\varphi\left(h_j^n(z)\right)-S_n\varphi\left(h_j^n(z')\right)\right| \leqslant A. \qquad \square$$

One can improve the above lemma as follows.

Lemma 5.5. *Let $0<\lambda<1$, $m\geqslant 1$ and U be a topological disk in $\mathcal{P}_M$ that does not contain any critical value of f^m. Then, for every $\epsilon>0$, there is $\delta>0$ such that*

$$\left|S_n\varphi\left(h_j^n(z)\right)-S_n\varphi\left(h_j^n(z')\right)\right| \leqslant \epsilon$$

for all $j\in J_n$ and for $z, z'\in U$ with $dist_{\chi,U}(z,z')<\delta$, where $dist_{\chi,U}$ is the internal chordal metric in U.

To construct a Gibbs state μ_φ we define a normalized Perron–Frobenius operator

$$\mathcal{N}_\varphi := \rho^{-1}\mathcal{L}_\varphi = e^{-c}\mathcal{L}_\varphi, \quad c>0,$$

and decompose it into:

$$\mathcal{N}_\varphi^n = e^{-nc}\mathcal{L}_\varphi^n = \mathcal{G}_\varphi^n+\mathcal{A}_\varphi^n+\mathcal{B}_\varphi^n, \quad n\geqslant m, \tag{5.6}$$

where

$$\begin{aligned}
\mathcal{G}_\varphi^n\psi(z) &= \sum_{j\in J_n}\psi\left(h_j^n(z)\right)\exp\left\{S_n\varphi\left(h_j^n(z)\right)-nc\right\},\\
\mathcal{B}_\varphi^n\psi(z) &= \sum_{j\in I_n\setminus J_n}\psi\left(h_j^n(z)\right)\exp\left\{S_n\varphi\left(h_j^n(z)\right)-nc\right\},\\
\mathcal{A}^n &= \mathcal{N}_\varphi^n-\mathcal{G}_\varphi^n-\mathcal{B}_\varphi^n.
\end{aligned}$$

Note that the operator $\mathcal{G}_\varphi^n$ is defined for exponentially shrinking inverse branches, $\mathcal{B}_\varphi^n$ corresponds to the branches defined in Lemma 5.2 and $\mathcal{A}_\varphi^n$ to the rest part of branches mentioned in Lemma 5.3.

To prove Theorem 4.1 we need the following uniform estimate on $\|\mathcal{N}_\varphi^n\|$, $n\in\mathbb{N}$.

Proposition 5.6. *There exists a constant $C_1>0$ such that all $n\geqslant 0$*

$$\left\|\mathcal{N}_\varphi^n\mathbb{1}\right\| \leqslant C_1.$$

Proof. It follows from Lemma 5.5 that there is a constant $c_1 \geqslant 1$ such that for all $z, z' \in U$ and $n \geqslant m$

$$(5.7) \qquad \mathcal{G}_\varphi^n \mathbb{1}(z) \leqslant c_1 \mathcal{G}_\varphi^n \mathbb{1}(z').$$

Since

$$(5.8) \qquad \int_U \mathcal{G}_\varphi^n \mathbb{1}\, d\nu \leqslant \int_U \mathcal{N}_\varphi^n \mathbb{1}\, d\nu = \int_U \mathbb{1}\, d\nu = 1$$

then there exists a point $y_0 \in U$ such that $\mathcal{G}_\varphi^n \mathbb{1}(y_0) \leqslant \frac{1}{\nu(U)}$. (5.7) and (5.8) imply

$$(5.9) \qquad \mathcal{G}_\varphi^n \mathbb{1}(z) \leqslant c_1$$

for all $z \in U$ and $n \geqslant m$. For any $n \geqslant m$, $z \in U$ and $H_k(z)$ defined in Lemma 5.3 we have

$$\begin{aligned}
(5.10) \qquad \mathcal{A}_\varphi^n \mathbb{1}(z) &= \sum_{k=m+1}^{n} \sum_{y \in H_k(z)} \exp\{S_k\varphi(y) - kc\} \mathcal{N}_\varphi^{n-k} \mathbb{1}(y) \\
&\leqslant \sum_{k=m+1}^{n} \sharp H_k(z) \sigma^k \left\| \mathcal{N}_\varphi^{n-k} \mathbb{1} \right\| \\
&\leqslant \kappa d \sigma^{m+1} \sum_{k=0}^{n-m-1} \sigma^k \left\| \mathcal{N}_\varphi^{n-m-1} \mathbb{1} \right\|.
\end{aligned}$$

Analogously for $n \geqslant m$, $z \in U$ and E_k defined in Lemma 5.2

$$\begin{aligned}
(5.11) \qquad \mathcal{B}_\varphi^n \mathbb{1}(z) &\leqslant \sum_{k=m+1}^{n} \sum_{i \in E_k,\, y = h_i^k(z)} \exp\{S_k\varphi(y) - kc\} \mathcal{N}_\varphi^{n-k} \mathbb{1}(y) \\
&\leqslant \sum_{k=m+1}^{n} \sharp E_k \sigma^k \left\| \mathcal{N}_\varphi^{n-k} \mathbb{1} \right\| \\
&\leqslant \lambda^m \sum_{k=m+1}^{n} \left(\frac{\sigma}{\lambda}\right)^k \left\| \mathcal{N}_\varphi^{n-k} \mathbb{1} \right\| \\
&\leqslant \frac{\sigma^{m+1}}{\lambda} \sum_{k=0}^{n-m-1} \left(\frac{\sigma}{\lambda}\right)^k \left\| \mathcal{N}_\varphi^{n-m-1-k} \mathbb{1} \right\|.
\end{aligned}$$

Define

$$C_1 := \max\left\{ \frac{c_1}{1 - \alpha\sigma^{m+1}},\ \left\| \mathcal{N}_\varphi^n \mathbb{1} \right\|,\ 0 \leqslant k \leqslant m \right\}.$$

Let $n > m$ and assume that $\|\mathcal{N}_\varphi^n \mathbb{1}\| \leqslant C_1$ for $k = 0, \dots, n-1$, then (5.10) and (5.11) yield

$$\mathcal{A}_\varphi^n \mathbb{1}(z) + \mathcal{B}_\varphi^n \mathbb{1}(z) \leqslant C_1 \sigma^{m+1}\left(\frac{\kappa d}{1-\sigma} + \frac{1}{\lambda - \sigma}\right) = \alpha \sigma^{m+1} C_1.$$

Therefore

$$\mathcal{N}_\varphi^n \mathbb{1}(z) = \mathcal{G}_\varphi^n \mathbb{1}(z) + \mathcal{A}_\varphi^n \mathbb{1}(z) + \mathcal{B}_\varphi^n \mathbb{1}(x) \leqslant C_1 + \alpha\sigma^{m+1} C_1 \leqslant C_1$$

for every $z \in U$. Since U is dense in $\mathcal{P}_M$ and by continuity of $\mathcal{N}_\varphi^n \mathbb{1}$ we obtain that $\|\mathcal{N}_\varphi^n \mathbb{1}\| \leqslant C_1$. □

The proof of next lemma is the same as Proposition 6.7 in [5].

Lemma 5.7. *There exists a constant $C_2 > 0$ such that*

$$\mathcal{N}_\varphi^n \mathbb{1}(z) \geqslant C_2$$

for all $n \geqslant 1$ and $z \in \mathcal{P}_M$.

Proof of Theorem 4.1. Let $z_0 \in J_f$ be a point such that $\sup(\varphi) < P_{z_0}(\varphi)$. Then Proposition 5.6 and Lemma 5.7 imply

$$C_2 \leqslant \rho^{-n} \mathcal{L}_\varphi^n \mathbb{1}(x) \leqslant C_1 \tag{5.12}$$

for all $z \in \mathcal{P}_M$ and $n \geqslant 1$, where $\rho = \exp(P_{z_0}(\varphi))$. Therefore $P_z(\varphi)$ is constant. Let m be any te^{φ}-conformal measure. Then $\mathcal{L}_\varphi^* m = tm$ and $\log(t) = \frac{1}{n}\int \mathcal{L}_\varphi^n \mathbb{1}\, dm$ for all $n \geqslant 1$. This together with (5.11) yields $t = \rho = e^{P(\varphi)}$. □

Proof of Theorem 4.2. Let m_φ be $\rho e^{-\varphi}$-conformal measure supported on J_f, where φ is a summable potential satisfying

$$\sup(\varphi) < \sup\{P_z(\varphi):\ z \in J_f\}.$$

The measure m_φ is a fixed point of the normalized dual operator $\mathcal{N}_\varphi^*$. To construct a Gibbs state μ_φ we have to find a density $h = \frac{d\mu_\varphi}{dm_\varphi}$. We construct h as a fixed point of $\mathcal{N}_\varphi$. For $z \in J_f$ define

$$\tilde{h}(z) = \liminf_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \mathcal{N}_\varphi^k \mathbb{1}(z)$$

and let $h_n = \frac{1}{n}\sum_{k=1}^{n} \mathcal{N}_\varphi^k \mathbb{1}$. Then

$$\mathcal{N}_\varphi(h_n) = h_n + \frac{1}{n}\left(\mathcal{N}_\varphi^{k+1}\mathbb{1} - \mathcal{N}_\varphi \mathbb{1}\right).$$

Fix $z \in J_f$ and choose a subsequence (h_{n_j}) which converges to $\tilde{h}$. Then $\mathcal{N}_\varphi(h_{n_j})(z) \to \tilde{h}(z)$. Using this one can prove that $\tilde{h}(z) \geqslant \mathcal{N}_\varphi(\tilde{h})(z)$ for all $z \in J_f$. Since also $\int \mathcal{N}_\varphi(\tilde{h})\, dm_\varphi = \int \tilde{h}\, dm_\varphi$ we can define

$$h := \frac{\tilde{h}}{\int \tilde{h}\, dm_\varphi}.$$

Then $d\mu_\varphi = h\, dm_\varphi$ defines a Gibbs state μ_φ equivalent to $\exp(P(\varphi) - \varphi)$-conformal measure m_φ. Uniqueness and ergodicity of μ_φ follows from [4]. Analogously as in [5] one can prove that μ_φ is supported on the conical subset of J_f. $\square$

Theorems 4.1 and 4.2 imply Theorem A.

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